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# *A Deduction and Demonstration of Taylor's Formula.*

BY W. H. ECHOLS.

The following method of deducing the formula for the expansion of  $f(x+h)$  in terms of ascending powers of  $h$  is of interest, because it does not require the assumption of the possibility of the series nor that it should be differentiable.

The determinant

$$\begin{vmatrix} fx & 1 & x & \dots & x^n \\ fa_0 & 1 & a_0 & \dots & a_0^n \\ \dots & \dots & \dots & \dots & \dots \\ fa_n & 1 & a_n & \dots & a_n^n \end{vmatrix} \quad (1)$$

vanishes for the  $n+1$  values of  $x, a_0 \dots a_n$ . Its first derivative vanishes for  $n$  values of  $x$  between these values, by Rolle's theorem,  $fx$  being a continuous function for the limits prescribed. Its second derivative vanishes for  $n-1$  values of  $x$  between these values, and so on, until evidently its  $n^{\text{th}}$  derivative vanishes for some value  $u$ , of  $x$ , which lies between the greatest and least of the values  $a_0 \dots a_n$ .

Let  $Fx$  represent the above determinant, and for brevity write

$$\zeta^{\dagger} = \zeta^{\dagger}(a_0 \dots a_n).$$

We then have

$$Fx = \zeta^{\dagger}fx + \phi x, \quad (2)$$

wherein  $\phi x$  is a rational integral function of the  $n^{\text{th}}$  degree.

This being so, we have

$$\begin{aligned} F(x+h) &= \zeta^{\dagger}f(x+h) + \phi(x+h) \\ &= \zeta^{\dagger}f(x+h) + \phi x + \frac{h^1}{1!} \phi'x + \dots + \frac{h^n}{n!} \phi^n x. \end{aligned} \quad (3)$$

Differentiate (2)  $n$  times and multiply these equations through respectively by  $h^r/r!$ , ( $r = 1 \dots n$ ), whence results

$$\begin{aligned} hFx &= \zeta^{\dagger} h f'x + h\phi'x, \\ &\dots\dots\dots \\ \frac{h^{n-1}}{(n-1)!} F^{n-1}x &= \zeta^{\dagger} \frac{h^{n-1}}{(n-1)!} f^{n-1}x + \frac{h^{n-1}}{(n-1)!} \phi^{n-1}x, \\ \frac{h^n}{n!} F^n u &= \zeta^{\dagger} \frac{h^n}{n!} f^n u + \frac{h^n}{n!} \phi^n u = 0. \end{aligned}$$

Subtracting these equations from (3), member by member, we obtain

$$\begin{aligned} f(x+h) - fx - \frac{h^1}{1!} f'x - \dots - \frac{h^{n-1}}{(n-1)!} f^{n-1}x - \frac{h^n}{n!} f^n u \\ = \frac{1}{\zeta^{\dagger}} \left[ F(x+h) - Fx - \frac{h^1}{1!} F'x - \dots - \frac{h^{n-1}}{(n-1)!} F^{n-1}x \right]. \quad (4) \end{aligned}$$

Since the  $a$ 's are arbitrary, we may shift them as we choose, so put  $a_0 = x$  and  $a_n = x+h$ , then  $Fx = 0$ , also  $F(x+h) = 0$ , and the second member of (4) becomes

$$- \frac{\frac{h^1}{1!} F'x + \dots + \frac{h^{n-1}}{(n-1)!} F^{n-1}x}{\zeta^{\dagger}(x, a_1 \dots a_{n-1}, x+h)},$$

which takes the indeterminate form  $0/0$  whenever  $a_1, \dots, a_{n-1} = x$ .

To evaluate the true form of this ratio when  $a_1, \dots, a_{n-1} = x$ , apply to the numerator and denominator the operator

$$\left( \frac{d}{da_1} \right)_{a_1=x}^1 \dots \left( \frac{d}{da_{n-1}} \right)_{a_{n-1}=x}^{n-1},$$

$\left( \frac{d}{da_r} \right)_{a_r=x}^r$  causes  $F^r x$  to vanish ( $r = 1 \dots n-1$ ), while

$$\left( \frac{d}{da_1} \right)_{a_1=x}^1 \dots \left( \frac{d}{da_{n-1}} \right)_{a_{n-1}=x}^{n-1} \zeta^{\dagger}(x, a_0, \dots, a_{n-1}, x+h) = (n-1)!! h^n.$$

Hence the true value of the ratio is zero and we have

$$f(x+h) = fx + \frac{h^1}{1!} f'x + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}x + \frac{h^n}{n!} f^n u. \quad (x < u < x+h)$$

The method of determining the ultimate ratio of an indeterminate form can be developed wholly independent of Taylor's formula (Todhunter's *Diff. Calculus*, p. 124); it seems, therefore, that the above analysis is free from objection.